

# On bounds of Tsallis relative entropy and an inequality for generalized skew information

Shigeru Furuichi<sup>1\*</sup>

<sup>1</sup>Department of Computer Science and System Analysis,  
College of Humanities and Sciences, Nihon University,  
3-25-40, Sakurajyousui, Setagaya-ku, Tokyo, 156-8550, Japan

**Abstract.** Quantum entropy and skew information play important roles in quantum information science. They are defined by the trace of the positive operators so that the trace inequalities often have important roles to develop the mathematical theory in quantum information science. In this paper, we study some properties for information quantities in quantum system through trace inequalities. Especially, we give upper bounds and lower bounds of Tsallis relative entropy, which is a one-parameter extension of the relative entropy in quantum system. In addition, we compare the known bounds and the new bounds, for both upper and lower bounds, respectively. We also give an inequality for generalized skew information by introducing a generalized correlation measure.

**Keywords :** Tsallis relative entropy, trace inequality, skew information, correlation measure and positive operators

**2000 Mathematics Subject Classification :** 15A45, 47A63 and 94A17

## 1 Introduction

Two important theorems in quantum information theory have been proved in [42] and [26, 43]. In the paper [42], it was shown the relation between quantum entropy (von Neumann entropy) [50] and source coding theorem in quantum system (non-commutative system). In the papers [26, 43], it was also shown the relation between the Holevo bound, which is considered to be mutual information in quantum information theory, and coding theorem for the classical-quantum channel. Especially, in these papers [26, 42, 43], the role of von Neumann entropy and Holevo bound in quantum information were clarified so that quantum information theory has been progressed for around fifteen years [35, 38]. Before such developments of quantum information, von Nuemann entropy and related entropies such as relative entropy [49] and mutual entropy [36] were studied in both direction from physics and mathematics [37, 51]. The study for a generalization on information entropy in classical system (commutative system) has a long history and we have many literatures [7, 39, 45]. See also [1, 8] and references therein. As for one of the generalizations of entropy, we have studied the Tsallis entropy and the Tsallis relative entropy in quantum system [11, 12, 13, 14, 15, 16, 17]. In the present paper, we study on the bounds of the Tsallis relative entropy which is a one-parameter extension of the Umegaki relative entropy.

As one of the mathematical studies on the topics about entropy theory, skew information [52, 53] and its concavity problem are famous. The concavity problem for skew information

---

\*E-mail: furuichi@chs.nihon-u.ac.jp

was generalized by F.J.Dyson, and it was proven by E.H.Lieb in [30]. It is also known that skew information presents the degree of noncommutativity between a certain quantum state represented by a density operator  $\rho$  (which is a positive operator with an unit trace) and an observable represented by self-adjoint operator  $H$ , therefore an uncertainty relation using skew information has been studied in [11, 18, 28, 31, 34, 32, 54]. See [24, 40, 41] for the original uncertainty relations which were represented by the trace inequalities and they show the uncertainty principle which is a fundamental concept in quantum mechanical physics. In the present paper, we give a trace inequality for a generalized skew information by introducing a generalized correlation measure.

## 2 Upper bounds of Tsallis relative entropy

Firstly we give some notation. We denote the generalized exponential function by  $\exp_\nu(x) \equiv (1 + \nu x)^{\frac{1}{\nu}}$  if  $1 + \nu x > 0$ , otherwise it is undefined and its inverse function (generalized logarithmic function) by  $\ln_\nu x \equiv \frac{x^\nu - 1}{\nu}$ , for  $\nu \in (0, 1]$  and  $x \geq 0$ . The functions  $\exp_\nu(x)$  and  $\ln_\nu x$  converge to  $e^x$  and  $\log x$  as  $\nu \rightarrow 0$ , respectively. Note that the definition of the generalized logarithmic function  $\ln_\nu(X)$  for a positive operator  $X$  is well-defined. In this paper, we define the generalized exponential function  $\exp_\nu(X)$  for a positive operator  $X$  by  $\exp_\nu(X) \equiv (I + \nu X)^{\frac{1}{\nu}}$ , if  $\text{Tr} \left[ (I + \nu X)^{\frac{1}{\nu}} \right] \in \mathbb{R}$ .

The Tsallis relative entropy in quantum system (noncommutative system) is defined for the positive operators in the following manner. For the study of entropies from mathematical viewpoint, the condition such as an unit trace is often relaxed. See p.274 of [5] or [11, 12]. See also [45, 46, 47, 48] for the original Tsallis entropy and its advances in statistical physics.

**Definition 2.1** *The Tsallis relative entropy is defined by*

$$D_\nu(X|Y) \equiv \frac{\text{Tr}[X - X^{1-\nu}Y^\nu]}{\nu} = \text{Tr}[X^{1-\nu}(\ln_\nu X - \ln_\nu Y)]$$

for positive operators  $X, Y$  and  $\nu \in (0, 1]$ .

We have the following proposition, which gives an upper bound of the Tsallis relative entropy.

**Proposition 2.2** ([11]) *For positive operators  $X, Y$  and  $\nu \in (0, 1]$ , the following inequality holds.*

$$D_\nu(X|Y) \leq -\text{Tr} \left[ X \ln_\nu \left( X^{-1/2} Y X^{-1/2} \right) \right]. \quad (1)$$

The further upper bound of the right hand side was given by T.Furuta in [21], with the generalized Kantorovich constant:

$$D_\nu(X|Y) \leq -\text{Tr} \left[ X \ln_\nu \left( X^{-1/2} Y X^{-1/2} \right) \right] \leq \left( \frac{1 - K(\nu, h)}{\nu} \right) \text{Tr}[X]^{1-\nu} \text{Tr}[Y]^\nu + D_\nu(X|Y),$$

where  $X^{1/2} \ln_\nu (X^{-1/2} Y X^{-1/2}) X^{1/2}$  is often called the Tsallis relative operator entropy for positive operators  $X, Y$  and  $K(\nu, h)$  is the generalized Kantorovich constant defined for  $\nu \in \mathbb{R}$  and  $h \in (0, \infty)$  with  $h \neq 1$ :

$$K(\nu, h) \equiv \frac{(h^\nu - h)}{(\nu - 1)(h - 1)} \left( \frac{(\nu - 1)}{\nu} \frac{h^\nu - 1}{(h^\nu - h)} \right)^\nu.$$

We note that  $K(2, \frac{M}{m}) = K(-1, \frac{M}{m}) = \frac{(M+m)^2}{4mM}$ , which is often called the Kantorovich constant. See [22, 23] for details. Since we have  $K(0, h) = K(1, h) = 1$  and  $\frac{dK(\nu, h)}{d\nu}|_{\nu=0} = -\log S(h)$ , the above inequalities recover the following inequalities (the first inequality below was originally given in [25]) :

$$U(X|Y) \leq -Tr \left[ X \log \left( X^{-1/2} Y X^{-1/2} \right) \right] \leq Tr[X] \log S(h) + U(X|Y),$$

in the limit  $\nu \rightarrow 0$ . Where  $U(X|Y) \equiv Tr[X(\log X - \log Y)]$  is the relative entropy introduced by Umegaki in [49],  $X^{1/2} \log(X^{-1/2} Y X^{-1/2}) X^{1/2}$  is the relative operator entropy introduced in [4, 10] and  $S(h) \equiv \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$  is the Specht's ratio [44], where  $h \equiv \frac{M_1 M_2}{m_1 m_2} > 1$  for  $0 < m_1 I \leq X \leq M_1 I$  and  $0 < m_2 I \leq Y \leq M_2 I$ . (See [21] for details.)

We also have the following proposition.

**Proposition 2.3** *For positive operators  $X, Y$  and  $\nu \in (0, 1]$ , the following inequality holds.*

$$D_\nu(X|Y) \leq \frac{Tr[(X - Y)_+]}{\nu}, \quad (2)$$

where  $A_+ \equiv \frac{1}{2}(A + |A|)$  and  $|A| \equiv (A^* A)^{1/2}$  for any operator  $A$ .

*Proof:* It immediately follows from the trace inequality proven by K.M.R.Audenaert et.al. in [2]:

$$Tr[A^s B^{1-s}] \geq \frac{1}{2} Tr[A + B - |A - B|] \quad (3)$$

for positive operators  $A, B$  and  $s \in [0, 1]$ . ■

From the above two propositions, we have two different upper bounds for the Tsallis relative entropy. It is quite natural to consider that we have the following inequality for positive operators  $X$  and  $Y$  and  $\nu \in (0, 1]$ :

$$-Tr \left[ X \ln_\nu \left( X^{-1/2} Y X^{-1/2} \right) \right] \leq \frac{Tr[(X - Y)_+]}{\nu}, \quad (4)$$

which is equivalent to

$$Tr[X \sharp_\nu Y] \geq \frac{1}{2} Tr[X + Y - |X - Y|], \quad (5)$$

where  $X \sharp_\nu Y \equiv X^{1/2} (X^{-1/2} Y X^{-1/2})^\nu X^{1/2}$  is  $\nu$ -power mean. Here we note that we have

$$Tr[X^{1-\nu} Y^\nu] \geq Tr[X \sharp_\nu Y]$$

from the inequality (1). However we have the counter-example for the inequality (5) in the following. We take  $\nu = 1/2$  and

$$X = \begin{pmatrix} 10 & 7 \\ 7 & 5 \end{pmatrix}, Y = \begin{pmatrix} 16 & 6 \\ 6 & 3 \end{pmatrix}.$$

Then we have

$$Tr[X \sharp_{1/2} Y] - \frac{1}{2} Tr[X + Y - |X - Y|] \simeq -0.510619.$$

Therefore the inequality (4) does not hold in general. That is, we can conclude that neither the inequality (1) nor the inequality (2) is uniformly better than the other.

### 3 Lower bounds of Tsallis relative entropy

We firstly note that the Tsallis relative entropy defined in Definition 2.1 is not always nonnegative. (If we impose on the condition such as an unit trace for two positive operators  $X$  and  $Y$ , then the Tsallis relative entropy has a nonnegativity.) Therefore it is natural to have an interest in the lower bound of the Tsallis relative entropy defined for two positive operators. We have the following proposition, which gives a lower bound of the Tsallis relative entropy.

**Proposition 3.1** ([11]) *For positive operators  $X, Y$  and  $\nu \in (0, 1]$ , the generalized Bogoliubov inequality holds.*

$$D_\nu(X|Y) \geq \frac{\text{Tr}[X] - (\text{Tr}[X])^{1-\nu}(\text{Tr}[Y])^\nu}{\nu} \quad (6)$$

If we take the limit  $\nu \rightarrow 0$ , Proposition 3.1 recovers the original Peierls-Bogoliubov inequality [3, 27]:

$$U(X|Y) \geq \text{Tr}[X (\log \text{Tr}[X] - \log \text{Tr}[Y])].$$

We also easily find that the right hand side in the inequality (6) is nonnegative, if we have the relation such that  $\text{Tr}[X] \geq \text{Tr}[Y]$ . We also have the following lower bound for the Tsallis relative entropy.

**Theorem 3.2** *For positive operators  $X, Y$  and  $\nu \in (0, 1]$ , if we have  $I \leq Y \leq X$ , then we have the following inequality*

$$D_\nu(X|Y) \geq \text{Tr} \left[ X^{1-\nu} \ln_\nu \left( Y^{-1/2} X Y^{-1/2} \right) \right]. \quad (7)$$

It is notable that the condition  $X \geq Y$  assures the nonnegativity of the right hand side in the inequality (7). To prove Theorem 3.2, we use the following lemmas.

**Lemma 3.3** ([19]) *For positive operators  $X, Y$  and  $\nu \in (0, 1]$ , we have*

$$\text{Tr}[\exp_\nu(X + Y)] \leq \text{Tr}[\exp_\nu(X) \exp_\nu(Y)]. \quad (8)$$

Here we give a slightly different version of a variational expression for the Tsallis relative entropy. It can be proven by the similar way to Theorem 2.1 in [13]. However we give the proof for the convenience of readers as to be a self-contained article.

**Lemma 3.4** *For any  $\nu \in (0, 1]$  and any  $d \in [0, \infty)$ , we have the following relations.*

(i) *If  $A$  and  $Y$  are positive operators, then we have*

$$d \ln_\nu \left( \frac{\text{Tr}[\exp_\nu(A + \ln_\nu Y)]}{d} \right) = \max \{ \text{Tr}[X^{1-\nu} A] - D_\nu(X|Y) : X \geq 0, \text{Tr}[X] = d \}.$$

(ii) *If  $X$  and  $B$  are positive operators with  $\text{Tr}[X] = d$ , then*

$$D_\nu(X|\exp_\nu(B)) = \max \left\{ \text{Tr}[X^{1-\nu} A] - d \ln_\nu \left( \frac{\text{Tr}[\exp_\nu(A + B)]}{d} \right) : A \geq 0 \right\}.$$

*Proof:* For the case of  $\nu = 1$ , it is trivial so that we assume  $\nu \in (0, 1)$ . Since we have  $\lim_{x \rightarrow 0} x \ln_\nu \frac{a}{x} = 0$  for  $\nu \in (0, 1)$ , it is also trivial for the case of  $X = 0$ , thus we assume  $X \neq 0$ .

(1) We define

$$F_\nu(X) \equiv \text{Tr}[X^{1-\nu}A] - D_\nu(X|Y)$$

for a positive operator  $X$  with  $\text{Tr}[X] = d < \infty$ . If we take the Schatten decomposition  $X = \sum_{j=1}^{\infty} \mu_j E_j$ , where all  $E_j$ , ( $j = 1, 2, \dots, \infty$ ) are projections of rank one with  $\sum_{j=1}^{\infty} E_j = I$  and  $\mu_j \geq 0$ , ( $j = 1, 2, \dots, \infty$ ) with  $\sum_{j=1}^{\infty} \mu_j = d$ , then we rewrite

$$F_\nu \left( \sum_{j=1}^{\infty} \mu_j E_j \right) = \sum_{j=1}^{\infty} \left\{ \mu_j^{1-\nu} \text{Tr}[E_j A] + \frac{1}{\nu} \mu_j^{1-\nu} \text{Tr}[E_j Y^\nu] - \frac{1}{\nu} \mu_j \text{Tr}[E_j] \right\}.$$

Then we have

$$\frac{\partial^2}{\partial \mu_j^2} F_\nu \left( \sum_{j=1}^{\infty} \mu_j E_j \right) = -\nu(1-\nu) \mu_j^{-\nu-1} \text{Tr} \left[ E_j \left( A + \frac{1}{\nu} Y^\nu \right) \right] \leq 0,$$

which means  $F_\nu$  is concave function. Thus we find  $F_\nu(X)$  attains its maximum at a certain positive operator  $X_0$  with  $\text{Tr}[X_0] = d$ . Then for any self-adjoint operators  $S$  with  $\text{Tr}[S] = 0$  (since for any  $t \in \mathbb{R}$ ,  $\text{Tr}[X_0 + tS] = d$  which is a condition on the positive operator defined on the domain of the function  $F_\nu$ ), there exists a positive operator  $X_0$  such that

$$0 = \frac{d}{dt} F_\nu(X_0 + tS)|_{t=0} = (1-\nu) \text{Tr} \left[ S(X_0^{-\nu} A + \frac{1}{\nu} X_0^{-\nu} Y^\nu) \right],$$

so that  $X_0^{-\nu} A + \frac{1}{\nu} X_0^{-\nu} Y^\nu = cI$  for  $c \in \mathbb{R}$ . Thus we have

$$X_0 = d \frac{\exp_\nu(A + \ln_\nu Y)}{\text{Tr}[\exp_\nu(A + \ln_\nu Y)]}.$$

by putting  $c = \frac{1}{\nu}$  and satisfying the condition  $\text{Tr}[X_0] = d$ . By the formulae  $\ln_\nu \frac{y}{x} = \ln_\nu y + y^\nu \ln_\nu \frac{1}{x}$  and  $\ln_\nu \frac{1}{x} = -x^{-\nu} \ln_\nu x$ , we have

$$\begin{aligned} F_\nu(X_0) &= d^{1-\nu} \frac{\text{Tr}[\{\exp_\nu(A + \ln_\nu Y)\}^{1-\nu} A]}{\text{Tr}[\exp_\nu(A + \ln_\nu Y)]^{1-\nu}} \\ &\quad - d^{1-\nu} \text{Tr} \left[ \frac{\{\exp_\nu(A + \ln_\nu Y)\}^{1-\nu}}{\text{Tr}[\exp_\nu(A + \ln_\nu Y)]^{1-\nu}} \left( \ln_\nu \left( \frac{\exp_\nu(A + \ln_\nu Y)}{\frac{1}{d} \text{Tr}[\exp_\nu(A + \ln_\nu Y)]} \right) - \ln_\nu Y \right) \right] \\ &= d^{1-\nu} \frac{\text{Tr}[\{\exp_\nu(A + \ln_\nu Y)\}^{1-\nu} A]}{\text{Tr}[\exp_\nu(A + \ln_\nu Y)]^{1-\nu}} - d^{1-\nu} \text{Tr} \left[ \frac{\{\exp_\nu(A + \ln_\nu Y)\}^{1-\nu}}{\text{Tr}[\exp_\nu(A + \ln_\nu Y)]^{1-\nu}} \{A + \ln_\nu Y \right. \\ &\quad \left. + \{\exp_\nu(A + \ln_\nu Y)\}^\nu \ln_\nu \left( \frac{d}{\text{Tr}[\exp_\nu(A + \ln_\nu(A + \ln_\nu Y))]} \right) - \ln_\nu Y \right\} \right] \\ &= -d^{1-\nu} \text{Tr}[\exp_\nu(A + \ln_\nu Y)]^\nu \ln_\nu \left( \frac{d}{\text{Tr}[\exp_\nu(A + \ln_\nu Y)]} \right) \\ &= d \ln_\nu \frac{\text{Tr}[\exp_\nu(A + \ln_\nu Y)]}{d}. \end{aligned}$$

(2) It follows from (1) that the functional

$$g(A) \equiv d \ln_\nu \left( \frac{\text{Tr}[\exp_\nu(A + B)]}{d} \right), \quad d \equiv \text{Tr}[X] < \infty$$

defined on the set of all positive operator is convex, due to triangle inequality on max. Now let  $A_0 = \ln_\nu X - B$ , and define

$$G_\nu(A) \equiv \text{Tr}[X^{1-\nu}A] - d \ln_\nu \left( \frac{\text{Tr}[\exp_\nu(A+B)]}{d} \right),$$

which is concave on the set of all positive operator. Then for any self-adjoint operators  $S$ , there exists a positive operator  $A_0$  such that

$$\begin{aligned} \frac{d}{dt} G_\nu(A_0 + tS)|_{t=0} &= \text{Tr}[X^{1-\nu}S] - d \left( \frac{\text{Tr}[X]}{d} \right)^{\nu-1} \frac{\text{Tr}[S(I + \nu \ln_\nu X)^{\frac{1-\nu}{\nu}}]}{d} \\ &= \text{Tr}[X^{1-\nu}S] - \text{Tr}[SX^{1-\nu}] = 0, \end{aligned}$$

using the formulae  $\frac{d}{dx} \ln_\nu(x) = x^{\nu-1}$  and  $\frac{d}{dx} \exp_\nu(x) = (1 + \nu x)^{\frac{1}{\nu}-1}$ . Therefore  $G_\nu(A)$  attains the maximum

$$\begin{aligned} G_\nu(A_0) &= \text{Tr}[X^{1-\nu}(\ln_\nu X - B)] - d \ln_\nu \left( \frac{\text{Tr}[\exp_\nu(\ln_\nu X - B + B)]}{d} \right) \\ &= \text{Tr}[X^{1-\nu}(\ln_\nu X - B)] - d \ln_\nu \left( \frac{\text{Tr}[X]}{d} \right) \\ &= \text{Tr}[X^{1-\nu}(\ln_\nu X - \ln_\nu \exp_\nu(B))] \\ &= D_\nu(X | \exp_\nu(B)). \end{aligned}$$

■

If we take  $d = 1$  and the limit  $\nu \rightarrow 0$ , then Lemma 3.4 recovers Lemma 2.1 in [25] for positive operators  $A$  and  $B$ . It is notable that the original variational expressions for the Umegaki relative entropy proved by F.Hiai and D.Petz in [25] holds for Hermitian matrices  $A$  and  $B$ . We are now in a position to prove Theorem 3.2.

*Proof of Theorem 3.2:* Putting  $B = \ln_\nu Y$  and  $A = \ln_\nu Y^{-1/2}XY^{-1/2}$  in (ii) of Lemma 3.4 under the assumption of  $I \leq Y \leq X$  which assures  $A \geq 0$  and  $B \geq 0$ , and then using Lemma 3.3, we have

$$\begin{aligned} D_\nu(X|Y) &= D_\nu(X | \exp_\nu(\ln_\nu Y)) \\ &= D_\nu(X | \exp_\nu(B)) \\ &\geq \text{Tr}[X^{1-\nu}A] - \text{Tr}[X] \ln_\nu \left( \frac{\text{Tr}[\exp_\nu(A+B)]}{\text{Tr}[X]} \right) \\ &\geq \text{Tr}[X^{1-\nu}A] - \text{Tr}[X] \ln_\nu \left( \frac{\text{Tr}[\exp_\nu(A) \exp_\nu(B)]}{\text{Tr}[X]} \right) \\ &= \text{Tr}[X^{1-\nu} \ln_\nu Y^{-1/2}XY^{-1/2}] - \text{Tr}[X] \ln_\nu \left( \frac{\text{Tr}[Y^{-1/2}XY^{-1/2}Y]}{\text{Tr}[X]} \right) \\ &= \text{Tr}[X^{1-\nu} \ln_\nu Y^{-1/2}XY^{-1/2}]. \end{aligned}$$

■

**Remark 3.5** (I) The trace inequality (7) is equivalent to the following trace inequality:

$$\text{Tr} \left[ X^{1-\nu} \left\{ X^\nu - Y^\nu + I - \left( Y^{-1/2}XY^{-1/2} \right)^\nu \right\} \right] \geq 0.$$

Therefore, if the following matrix inequality:

$$X^\nu - Y^\nu + I - \left(Y^{-1/2}XY^{-1/2}\right)^\nu \geq 0 \quad (9)$$

holds, then the trace inequality (7) immediately holds. However the matrix inequality (9) does not hold in general, since we have the following counter-examples.

(i) If we take  $\nu = 1$  and

$$X = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}, Y = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

satisfying the condition  $I \leq Y \leq X$  (which is the assumption of Proposition 3.2), then one of the eigenvalues of the Hermitian matrix  $X - Y + I - Y^{-1/2}XY^{-1/2}$  takes a negative value.

(ii) If we take  $\nu = 1$  and

$$X = \frac{1}{9} \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}, Y = \frac{1}{3} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

satisfying the condition  $X \leq Y \leq I$ , then one of the eigenvalues of the Hermitian matrix  $X - Y + I - Y^{-1/2}XY^{-1/2}$  takes a negative value.

(II) Our next concern moves to the assumption of Proposition 3.2. We easily find that a counter-example for the trace inequality (7), in the case that our assumption  $I \leq Y \leq X$  is not satisfied. For example, if we take  $\nu = 1$  and

$$X = \frac{1}{15} \begin{pmatrix} 10 & -3 \\ -3 & 10 \end{pmatrix}, Y = \frac{1}{10} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

which does not satisfy the assumption  $I \leq Y \leq X$  (but satisfy  $0 < Y \leq X \leq I$ ), then

$$\text{Tr}[X - Y + I - Y^{-1/2}XY^{-1/2}] \simeq -20.9667.$$

Thus the inequality (7) does not hold in general for arbitrary positive operators  $X$  and  $Y$ . From (I) and (II), we may claim that Proposition 3.2 is not a trivial result.

Closing this section, we give a comment on the comparison of two lower bounds for the Tsallis relative entropy. Under the condition  $I \leq Y \leq X$ , we may have a conjecture such as

$$\text{Tr} \left[ X^{1-\nu} \ln_\nu \left( Y^{-1/2}XY^{-1/2} \right) \right] \geq \frac{\text{Tr}[X] - (\text{Tr}[X])^{1-\nu}(\text{Tr}[Y])^\nu}{\nu} \quad (10)$$

for positive operators  $X$  and  $Y$  and  $\nu \in (0, 1]$ . The inequality (10) is equivalent to the following inequality

$$\text{Tr} \left[ X^{1-\nu} \left( Y^{-1/2}XY^{-1/2} \right)^\nu \right] + (\text{Tr}[X])^{1-\nu}(\text{Tr}[Y])^\nu \geq \text{Tr}[X^{1-\nu}] + \text{Tr}[X]. \quad (11)$$

Here we take two positive definite matrices

$$X = \begin{pmatrix} 10 & 5 \\ 5 & 5 \end{pmatrix}, Y = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

satisfying the condition  $I \leq Y \leq X$ . Then for  $\nu = 0.1$ , we have

$$\text{Tr} \left[ X^{1-\nu} \left( Y^{-1/2}XY^{-1/2} \right)^\nu \right] + (\text{Tr}[X])^{1-\nu}(\text{Tr}[Y])^\nu - (\text{Tr}[X^{1-\nu}] + \text{Tr}[X]) \simeq 0.508133.$$

For  $\nu = 0.9$ , we also have

$$\text{Tr} \left[ X^{1-\nu} \left( Y^{-1/2} X Y^{-1/2} \right)^\nu \right] + (\text{Tr}[X])^{1-\nu} (\text{Tr}[Y])^\nu - (\text{Tr}[X^{1-\nu}] + \text{Tr}[X]) \simeq -1.1696.$$

Therefore the inequality (10) does not hold in general. That is, we can conclude that neither the inequality (6) nor the inequality (7) is uniformly better than the other. This result supports that our Theorem 3.2 is meaningful, in the sense of the comparison with Proposition 3.1.

## 4 An inequality for a generalized skew information

The uncertainty principle is a fundamental concept in quantum mechanical physics. It is represented by the famous Heisenberg uncertainty relation such as a trace inequality [24]:

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4}|\text{Tr}[\rho[A, B]]|^2 \quad (12)$$

for a quantum state  $\rho$  and two observables  $A$  and  $B$ . Where the variance for a quantum state  $\rho$  and an observable  $H$  is defined by  $V_\rho(H) \equiv \text{Tr}[\rho(H - \text{Tr}[\rho H]I)^2] = \text{Tr}[\rho H^2] - \text{Tr}[\rho H]^2$ . The further strong result was given by Schrödinger [41]:

$$V_\rho(A)V_\rho(B) - |\text{Re}\{Cov_\rho(A, B)\}|^2 \geq \frac{1}{4}|\text{Tr}[\rho[A, B]]|^2, \quad (13)$$

where the covariance is defined by  $Cov_\rho(A, B) \equiv \text{Tr}[\rho(A - \text{Tr}[\rho A]I)(B - \text{Tr}[\rho B]I)]$ . Due to its importance in quantum physics, the uncertainty relation has been studied by many researchers. Especially, some important results have been studied in the relation to the skew information representing a quantum uncertainty from the viewpoints of quantum information science. Here we firstly review about it. As it has been shown in [29, 34, 55], we do not have the uncertainty relation type inequality for the Wigner-Yanase skew information [52]:

$$I_\rho(H) \equiv \frac{1}{2}\text{Tr} \left[ (i[\rho^{1/2}, H_0])^2 \right] = \text{Tr}[\rho H^2] - \text{Tr}[\rho^{1/2} H \rho^{1/2} H], \quad (14)$$

where  $H_0 \equiv H - \text{Tr}[\rho H]I$  for a density operator  $\rho$  and an observable  $H$ . That is, the following trace inequality did not hold in general [29, 34, 55]:

$$I_\rho(A)I_\rho(B) \geq \frac{1}{4}|\text{Tr}[\rho[A, B]]|^2 \quad (15)$$

for a density operator  $\rho$  and observables  $A$  and  $B$ . Where  $[X, Y] \equiv XY - YX$  is a commutator. The counter example was given as follows.

**Counter-example 4.1 ([55])** *We take*

$$\rho = \frac{1}{4} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

*then we have  $I_\rho(A)I_\rho(B) = \left(1 - \frac{\sqrt{3}}{2}\right)^2$  and  $\frac{1}{4}|\text{Tr}[\rho[A, B]]|^2 = \frac{1}{4}$ . Therefore the inequality (15) does not hold in general.*

As a one-parameter generalization, the Wigner-Yanase-Dyson skew information was defined by

$$\begin{aligned} I_{\rho, \alpha}(H) &\equiv \frac{1}{2}\text{Tr} [(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])] \\ &= \text{Tr}[\rho H^2] - \text{Tr}[\rho^\alpha H \rho^{1-\alpha} H], \quad \alpha \in [0, 1] \end{aligned} \quad (16)$$



where  $H_0 \equiv H - \text{Tr}[\rho H]I$  for a density operator  $\rho$  and an observable  $H$ .

In [32], S.Luo introduced a new quantity such as

$$U_\rho(H) \equiv \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_\rho(H))^2} \quad (17)$$

for a density operator  $\rho$  and an observable  $H$ . Then he succeeded to establish the uncertainty relation type inequality as follows:

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4}|\text{Tr}[\rho[A, B]]|^2 \quad (18)$$

for a density operator  $\rho$  and observables  $A$  and  $B$ . He also introduced the quantity associated to Wigner-Yanase skew information,

$$J_\rho(H) \equiv \frac{1}{2}\text{Tr} \left[ \left( i \left\{ \rho^{1/2}, H_0 \right\} \right)^2 \right],$$

where the anti-commutator is defined by  $\{X, Y\} \equiv XY + YX$  for any operator  $X$  and  $Y$ . Then we have the relation  $U_\rho(H) = \sqrt{I_\rho(H)J_\rho(H)}$  and  $V_\rho(H) = \frac{1}{2}(I_\rho(H) + J_\rho(H))$ . Therefore, Luo's inequality (18) refines Heisenberg's one (12), since  $U_\rho(H) \leq V_\rho(H)$ .

K.Yanagi recently gave the generalization of the inequality (18) as follows [54]:

$$U_{\rho, \alpha}(A)U_{\rho, \alpha}(B) \geq \alpha(1 - \alpha)|\text{Tr}[\rho[A, B]]|^2 \quad (19)$$

for  $\alpha \in [0, 1]$ , a density operator  $\rho$  and observables  $A$  and  $B$ . Where  $U_{\rho, \alpha}(H)$  was defined by

$$U_{\rho, \alpha}(H) \equiv \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_{\rho, \alpha}(H))^2} \quad (20)$$

for a density operator  $\rho$  and an observable  $H$ .

In addition, quite recently, we gave the Schrödinger uncertainty relation for mixed states in [18]:

$$U_\rho(A)U_\rho(B) - |\text{Re}\{Corr_\rho(A, B)\}|^2 \geq \frac{1}{4}|\text{Tr}[\rho[A, B]]|^2, \quad (21)$$

where the correlation measure is defined for arbitrary operators  $X$  and  $Y$  by

$$Corr_\rho(X, Y) \equiv \text{Tr}[\rho X^*Y] - \text{Tr}[\rho^{1/2}X^*\rho^{1/2}Y].$$

On the other hand, S.Luo showed the trace inequality representing the relation between the original Wigner-Yanase skew information and the correlation measure in [32]:

$$I_\rho(A)I_\rho(B) \geq |\text{Re}\{Corr_\rho(A, B)\}|^2 \quad (22)$$

for a density operator  $\rho$  and two observables  $A$  and  $B$ .

It is remarkable that, if a quantum state (density operator)  $\rho$  is a pure state (i.e.,  $\rho^2 = \rho$ ), then the inequality (22) recovers

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4}|\text{Tr}[\rho\{A, B\}]|^2,$$

since  $I_\rho(H) = V_\rho(H)$  and  $Corr_\rho(A, B) = Cov_\rho(A, B)$  if  $\rho$  is a pure state. Where the covariance is defined by  $Cov_\rho(A, B) \equiv \text{Tr}[\rho AB] - \text{Tr}[\rho A]\text{Tr}[\rho B]$ .

In addition, defining a one-parameter extended correlation measure for  $\alpha \in [0, 1]$  and arbitrary operators  $X$  and  $Y$  by

$$Corr_{\rho, \alpha}(X, Y) \equiv \text{Tr}[\rho X^*Y] - \text{Tr}[\rho^\alpha X^*\rho^{1-\alpha}Y].$$

we have the following inequality:

$$I_{\rho,\alpha}(A)I_{\rho,\alpha}(B) \geq |\operatorname{Re}\{ \operatorname{Corr}_{\rho,\alpha}(A, B) \}|^2. \quad (23)$$

for a density operator  $\rho$ , two observables  $A, B$  and  $\alpha \in [0, 1]$ , putting  $\varepsilon = 0$  in Theorem III.4 of [55].

If we take  $\alpha = \frac{1}{2}$ , then the inequality (23) recovers the inequality (22).

To give the further generalized trace inequality, we give the following definition.

**Definition 4.2** Let  $f$  and  $g$  be the operator monotone functions. Let  $(f, g)$  be a monotonic pair. Where  $(f, g)$  is called a monotonic pair if  $(f(a) - f(b))(g(a) - g(b)) \geq 0$  for any  $a, b \in D \subset \mathbb{R}$ , for two functions  $f$  and  $g$  on the domain  $D \subset \mathbb{R}$ . For a density operator  $\rho$  and an observable  $H$ , we define  $(f, g)$ -skew information  $I_{\rho,(f,g)}(H)$  by

$$I_{\rho,(f,g)}(H) \equiv \frac{1}{2} \operatorname{Tr} [(i[f(\rho), H_0]) (i[g(\rho), H_0])], \quad (24)$$

where  $H_0 \equiv H - \operatorname{Tr}[\rho H]I$ . For a density operator  $\rho$  and any operators  $X, Y$ , we also define  $(f, g)$ -correlation measure  $\operatorname{Corr}_{\rho,(f,g)}(X, Y)$  by

$$\operatorname{Corr}_{\rho,(f,g)}(X, Y) \equiv \operatorname{Tr} [f(\rho)g(\rho)X^*Y] - \operatorname{Tr} [f(\rho)X^*g(\rho)Y]. \quad (25)$$

Then we can prove the following theorem.

**Theorem 4.3** For  $(f, g)$ -skew informations  $I_{\rho,(f,g)}(A)$ ,  $I_{\rho,(f,g)}(B)$  and  $(f, g)$ -correlation measure  $\operatorname{Corr}_{\rho,(f,g)}(A, B)$ , we have

$$I_{\rho,(f,g)}(A)I_{\rho,(f,g)}(B) \geq |\operatorname{Re}\{ \operatorname{Corr}_{\rho,(f,g)}(A, B) \}|^2. \quad (26)$$

*Proof:* For  $\operatorname{Corr}_{\rho,(f,g)}(X, Y)$ , we have the following properties, that is,  $\operatorname{Corr}_{\rho,(f,g)}(X, Y)$  is a sesquilinear form and Hermitian and  $\operatorname{Corr}_{\rho,(f,g)}(A, A)$  has the nonnegativity for a self-adjoint operator  $A$ , since  $(f, g)$  is a monotonic pair [6, 9]. Then for self-adjoint operators  $A$  and  $B$ , we have for any  $t \in \mathbb{R}$

$$\begin{aligned} 0 &\leq \operatorname{Corr}_{\rho,(f,g)}(tA + B, tA + B) \\ &= \operatorname{Tr}[f(\rho)g(\rho)(tA + B)(tA + B)] - \operatorname{Tr}[f(\rho)(tA + B)g(\rho)(tA + B)] \\ &= (\operatorname{Tr}[f(\rho)g(\rho)A^2] - \operatorname{Tr}[f(\rho)Ag(\rho)A]) t^2 + 2\operatorname{Re}(\operatorname{Tr}[f(\rho)g(\rho)AB] - \operatorname{Tr}[f(\rho)Ag(\rho)B]) t \\ &\quad + (\operatorname{Tr}[f(\rho)g(\rho)B^2] - \operatorname{Tr}[f(\rho)Bg(\rho)B]). \end{aligned}$$

Thus we have

$$\begin{aligned} &|\operatorname{Re}\{ \operatorname{Tr}[f(\rho)g(\rho)AB] - \operatorname{Tr}[f(\rho)Ag(\rho)B] \}|^2 \\ &\leq (\operatorname{Tr}[f(\rho)g(\rho)A^2] - \operatorname{Tr}[f(\rho)Ag(\rho)A]) (\operatorname{Tr}[f(\rho)g(\rho)B^2] - \operatorname{Tr}[f(\rho)Bg(\rho)B]), \end{aligned}$$

which implies the theorem, since we have

$$I_{\rho,(f,g)}(H) = \operatorname{Tr}[f(\rho)g(\rho)H^2] - \operatorname{Tr}[f(\rho)Hg(\rho)H].$$

■

Theorem 4.3 recovers the inequality (23), putting  $f(x) = x^\alpha$  and  $g(x) = x^{1-\alpha}$  for  $\alpha \in [0, 1]$ . We also have the following corollary.



- [3] N.Bebiano, J. da Providência, Jr. and R.Lemos, Matrix inequalities in statistical mechanics, *Linear Algebr. Appl.* Vol.376(2004),pp.265-273.
- [4] V.P.Belavkin and P.Staszewski,  $C^*$ -algebraic generalization of relative entropy and entropy, *Ann. Inst. Henri Poincaré, Sec. A*, Vol.37(1982), pp.51-58.
- [5] R.Bhatia, *Matrix Analysis*, Springer, 1997.
- [6] J.-C.Bourin, Some inequalities for norms on matrices and operators, *Linear Alg.Appl.*, Vol.292(1999), pp.139-154.
- [7] I. Csiszár, Axiomatic characterizations of information measures, *Entropy*, Vol.10(2008), pp.261-273.
- [8] B.Ebanks, P.Sahoo and W.Sander, *Characterizations of information measure*, World Scientific, 1998.
- [9] J.I.Fujii, A trace inequality arising from quantum information theory, *Linear Alg. Appl.*, Vol.400(2005), pp.141-146.
- [10] J.I.Fujii and E.Kamei, Relative operator entropy in noncommutative information theory, *Math. Japonica*, Vol.34(1989), pp.341-348.
- [11] S.Furuichi, K.Yanagi and K.Kuriyama, Fundamental properties of Tsallis relative entropy, *J.Math.Phys.*, Vol.45(2004), pp.4868-4877.
- [12] S.Furuichi, K.Yanagi and K.Kuriyama, A note on operator inequalities of Tsallis relative operator entropy, *Linear Alg. Appl.*, Vol.407(2005), pp.19-31.
- [13] S.Furuichi, Trace inequalities in nonextensive statistical mechanics, *Linear Alg.Appl.*, Vol.418(2006), pp.821-827.
- [14] S.Furuichi, A note on a parametrically extended entanglement-measure due to Tsallis relative entropy, *INFORMATION*, Vol.9,No.6,pp.837-844(2006).
- [15] S.Furuichi, Tsallis entropies and their theorems, properties and applications, pp.1-86, in the book “Aspects of Optical Sciences and Quantum Information”, Research Signpost, Edited by M.Abdel-Aty, 2007.
- [16] S.Furuichi, Matrix trace inequalities on Tsallis entropies, *Journal of inequalities in pure and applied mathematics*, Vol.9(2008), Issue 1, Article 1, 7pp.
- [17] S.Furuichi, A review of the mathematical properties of the Tsallis entropies, *AIP Conference Proceedings: Current Themes in Engineering Science 2007: Selected Presentations at the World Congress on Engineering-2007*, Vol.1045(1), pp.11 - 20, (September 9, 2008).
- [18] S.Furuichi, Schrödinger uncertainty relation with Wigner-Yanase skew information, to appear in *Phys.Rev.A*.
- [19] S.Furuichi and M.Lin, A matrix trace inequality and its application, *Linear Alg. Appl.* Vol.433(2010), pp.1324-1328.
- [20] S.Furuichi, K.Yanagi and K.Kuriyama, Trace inequalities on a generalized Wigner-Yanase skew information, *J.Math.Anal.Appl.*, Vol.356(2009), pp.179-185.

- [21] T.Furuta, Reverse inequalities involving two relative operator entropies and two relative entropies, *Linear Alg.Appl.*, Vol. 403(2005), pp. 24-30.
- [22] T.Furuta, *Invitation to linear operators*, Taylor and Francis, 2001.
- [23] T.Furuta, J.Mićić Hot, J.Rečarić and Y.Seo, Mond-Pečarić method in operator inequalities, *Element*, Zagreb, 2005.
- [24] W.Heisenberg, Über den anschaulichen Inhalt der quantummechanischen Kinematik und Mechanik, *Zeitschrift für Physik*, Vol.43(1927), pp.172-198.
- [25] F.Hiai and D.Petz, The Golden-Thompson trace inequality is complemented, *Linear Alg. Appl.*, Vol.181(1993), pp.153-185.
- [26] A.S.Holevo, The capacity of quantum channel with general signal states, *IEEE.Trans.IT*, Vol.44(1998), pp.269-273.
- [27] K.Huang, *Statistical Mechanics*, John Wiley and Sons, New York, 1987.
- [28] C.K.Ko, Comments on conjectures of trace inequalities on a generalized Wigner-Yanase skew information, *J.Math.Anal.Appl.*, Vol.369(2010), pp.164-167.
- [29] H.Kosaki, Matrix trace inequality related to uncertainty principle, *International Journal of Mathematics*, Vol.16(2005), pp.629-646.
- [30] E.H.Lieb, Convex trace functions and the Wigner-Yanase-Dyson conjecture, *Adv. Math.*, Vol.11(1973), pp.267-288.
- [31] M.Lin, On a conjecture of S.Furuichi et al., personal communications.
- [32] S.Luo, Heisenberg uncertainty relation for mixed states, *Phys.Rev.A*, Vol.72(2005), 042110.
- [33] S. Luo, Quantum versus classical uncertainty, *Theor. Math. Phys.* Vol.143(2005), pp.681-688.
- [34] S.Luo and Q.Zhang, On skew information, *IEEE Trans. Information Theory*, Vol.50(2004), pp.1778-1782, and Correction to "On skew information", *IEEE Trans. Information Theory*, Vol.51(2005), p.4432.
- [35] M.A.Nielsen and I.L.Chuang, *Quantum computation and quantum information*, Cambridge University Press, 2000.
- [36] M.Ohya, On compound state and mutual information in quantum information theory, *IEEE.Trans.Information Theory*, Vol.29(1983), pp.770-774.
- [37] M.Ohya and D.Petz, *Quantum entropy and its use*, Springer, 1993.
- [38] D.Petz, *Quantum information theory and quantum statistics*, Springer, 2008.
- [39] A.Rényi, On measures of entropy and information, in *Proc. 4th Berkeley Symp., Mathematical and Statistical Probability*, Berkeley, CA: Univ. Calif. Press, Vol. 1(1961), pp. 547-561.
- [40] H.P.Robertson, The uncertainty principle, *Phys.Rev.*, Vol.34(1929), pp.163-164.
- [41] E.Schrödinger, About Heisenberg uncertainty relation, *Proc.Prussian Acad.Sci., Phys.Math. Section*, Vol.XIX(1930), p.293.

- [42] B.Schumacher, Quantum coding, Phys.Rev.A, Vol.51(1995), pp.2738-2747.
- [43] B.Schumacher and M.D.Westmoreland, Sending classical information via noisy quantum channel, Phys.Rev.A, Vol.56(1997), pp.131-138.
- [44] W.Specht, Zer Theorie der elementaren Mittel, Math.Z.,Vol.74(1960), pp.91-98.
- [45] C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, J.Stat. Phys., Vol. 52(1988), pp. 479-487.
- [46] C. Tsallis et al. In: S. Abe and Y. Okamoto, Editors, Nonextensive Statistical Mechanics and its Applications, Springer, 2001. See also the comprehensive list of references at <http://tsallis.cat.cbpf.br/biblio.htm>.
- [47] C.Tsallis, Introduction to Nonextensive Statistical Mechanics: Approaching a Complex World, Springer, 2009.
- [48] C. Tsallis, Entropy, in Encyclopedia of Complexity and Systems Science, Springer, Berlin, 2009.
- [49] H.Umegaki, Conditional expectation in an operator algebra, IV (entropy and information),Kodai Math.Sem.Rep., Vol.14(1962), pp.59-85.
- [50] J.von Neumann,Thermodynamik quantenmechanischer Gesamtheiten,Göttinger Nachrichten,pp.273-291(1927).
- [51] A.Wehrl, General properties of entropy, Rev.Mod.Phys.,Vol.50(1978), pp.221-260.
- [52] E.P.Wigner and M.M.Yanase, Information content of distribution, Proc.Nat.Acad.Sci. U.S.A., Vol.49(1963), pp.910-918.
- [53] E.P.Wigner and M.M.Yanase, On the positive semidefinite nature of certain matrix expression, Canad. J. Math.,Vol.16(1964),pp.397-406.
- [54] K.Yanagi, Uncertainty relation on Wigner-Yanase-Dyson skew information, J.Math.Anal.Appl.,Vol.365(2010),pp.12-18.
- [55] K.Yanagi, S.Furuichi and K.Kuriyama, A generalized skew information and uncertainty relation, IEEE Trans. Information Theory, Vol.51(2005),pp.4401-4404.
- [56] K.Yanagi, S.Furuichi and K.Kuriyama, On generalized skew information, Proc. of 2005 Symp. on Appl. Func. Anal., pp.113-123.